

Plugging the Gauge Fixing into the Lagrangian

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Abstract

A complete analysis of the consequences of introducing a set of holonomic gauge fixing constraints (to fix the dynamics) into a singular Lagrangian is performed. It is shown in general that the dynamical system originated from the reduced Lagrangian erases all the information regarding the first class constraints of the original theory, but retains its second class. It is proved that even though the reduced Lagrangian can be singular, it never possesses any gauge freedom. As an application, the example of $n \cdot A = 0$ gauges in electromagnetism is treated in full detail.

PACS numbers: 0420-q, 0420Fy

1 Introduction

Take the Lagrangian for electromagnetism, $L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. L is singular because its Hessian matrix with respect to the velocities is singular. Now consider a generic axial gauge $n \cdot A = 0$, with $n_0 \neq 0$, and plug this constraint into the Lagrangian to eliminate A^0 . We end up with a reduced Lagrangian L_R which turns out to be regular and that can be used to define the propagator. The question is: is this procedure (plugging the constraint into the Lagrangian) correct?. Or put it in another way: what is the relation between the dynamics defined by L and that by L_R ?. Also: what happened to the Gauss Law $\nabla \cdot \mathbf{E} = 0$, which is a constraint for L but is no longer present for L_R ?

In this paper we will give an answer to these questions by studying the relations between L and L_R in the general case of a Constrained Dynamical System (CDS). There is sometimes a bit of confusion when using these words. Here we mean dynamical systems defined through a Lagrangian¹, that happens to be singular (i.e.: when the Hessian matrix of the Lagrangian with respect to the velocities is singular or, equivalently, when the Legendre transformation from velocity space to phase space is not invertible). Constraints naturally appear in the formalism as a consequence of the equations of motion –if we are in velocity space– or due to the fact that the Legendre transformation is not invertible –if we are in phase space. In phase space, the existence of constraints is compulsory, in velocity space not. The confusion can arise when one considers regular (non-singular) Dynamical Systems which are deliberately constrained to describe the motions in an *ad hoc* given surface. This surface can be either in configuration space (holonomic constraints) or in tangent space or phase space. In the holonomic case, one usually deals with these systems with the theory of Lagrange multipliers [1] [2] [3] [4], which is physically based on D'Alembert's principle. Let us notice that in general the presence of these *ad hoc* constraints amounts to a change of the original equations of motion.

Dirac [5][6], in his pioneering work, studied CDS in order to get a Hamiltonian formulation of gauge theories, including General Relativity (GR). As a general covariant theory, GR contains some gauge transformations (i.e.: symmetries that depend upon arbitrary functions of space-time. In the case of GR they are the spacetime diffeomorphisms) and it turns out that the price for having this kind of transformations in the formalism is that the Lagrangian must necessarily be singular. In fact, not only GR, but the most important quantum field theories, and also string theory, have also room for gauge transformations. This fact makes CDS a central issue in the modern study

¹The canonical formalism is built out of it by using the Legendre transformation.

of Dynamical Systems.

The existence of gauge transformations means that there are unphysical degrees of freedom in the formalism. This is also reflected in the fact that there is some arbitrariness in the dynamics, since to a given set of initial conditions there correspond several –actually, infinite– solutions of the equations of motion, which are related among themselves through gauge transformations. To get rid of these transformations, i.e., to quotient out the spurious gauge degrees of freedom, we must somewhat reduce the dimensionality of tangent space or phase space (depending upon we are working in Lagrangian or canonical formalism). One way of doing that, which proves convenient in varied circumstances, is by *ad hoc* introducing a new set constraints in order to eliminate the unphysical degrees of freedom. These constraints are called *Gauge Fixing* (GF) constraints and its role is twofold [7] [8]: fixing the dynamics and setting the physically inequivalent initial conditions. Notice that now we are introducing *ad hoc* constraints into the formalism as it can be done for regular theories. The difference is that now we do not intend to modify the dynamics, but rather to fix it by selecting one specific dynamics from the –gauge related– family of possible dynamics described by the equations of motion.

In this paper we will consider the GF procedure to fix the dynamics ² of a CDS in the case of an holonomic GF (i.e.: constraints defined in configuration space \mathcal{Q}). This is obviously the simplest case and it allows for both Hamiltonian and Lagrangian analysis. We find it instructive to compare the role of these *ad hoc* GF constraints introduced in a CDS with the role –dictated by D’Alembert principle– of the *ad hoc* constraints introduced in a regular Dinamical System. The geometric version of D’Alembert principle is: The holonomic constraints define a reduced configuration space \mathcal{Q}_R which has a natural injective map to \mathcal{Q} , $i : \mathcal{Q}_R \longrightarrow \mathcal{Q}$. i is naturally lifted to the tangent map between the tangent bundles, $i' : T\mathcal{Q}_R \longrightarrow T\mathcal{Q}$. Then, the new dynamics is defined in the velocity space $T\mathcal{Q}_R$ through a Lagrangian L_R which is the image of L (the original Lagrangian) under the pullback i'^* of i' . In plain words this means that we get L_R just by substituting the constraints into L .

This is standard theory for constrained regular Dinamical Systems, of course. But some questions arise when we try to use the same mechanism in order to fix the dynamics (through GF holonomic constraints) of a CDS. Now, in the singular case, we are lacking of any physical principle –like D’Alembert’s– to justify this procedure, but since it is available, it is worth to explore it. The question is: Can we proceed in the

²this is only a part of the whole GF procedure. The second part, as we have just said, consists in fixing the initial conditions.

same lines as it is done in the regular case, i.e.: to produce a reduced Lagrangian which dictates what the dynamics shall be?. In other words: it is correct to substitute the constraints into the lagrangian in order to get the correct gauge fixed dynamics? Is there any loss of information of the original theory if we proceed this way?. Our answer will be that the procedure is correct but, since there really is a loss of information, it must be supplemented with the addition of a specific subset of the primary Lagrangian constraints of the original theory.

Throghout the paper we will work mainly in tangent space but we will turn to the canonical formalism when we find it convenient. Some standard mathematical conditions for the Lagrangian are assumed, namely: the rank of the Hessian matrix is constant and –in Hamiltonian picture– no second class constraint can become first class through the stabilization algorithm.

In section 2 we will briefly consider the reduced Lagrangian formulation of the system. This formulation is equivalent to the extended one when use is made of the Lagrange multipliers. We will show in particular that the Lagrange multipliers for a CDS GF constraints are combinations of the original primary Lagrangian constraints of the theory.

The main result of this paper is (section 3) the following: Once the holonomic GF constraints to fix the dynamics are introduced, a) The reduced Lagrangian is singular if and only if the original theory has Hamiltonian second class constraints, and b) there is no gauge freedom for the reduced Lagrangian (i.e.: it has no first class Hamiltonian constraints).

In section 4 we illustrate our results with some examples. Comments and conclusions are given in section 5.

2 formalism in the reduced velocity space

Let us first consider a time-independent Lagrangian $L(q, \dot{q})$ in $T\mathcal{Q}$, where $q = \{q^i, i = 1, \dots, n\}$ are local coordinates for a point in a n -dimensional configuration space \mathcal{Q} . At this moment L can be either regular or singular.

Let us introduce a set of –independent– holonomic constraints $f_\mu(q) = 0$, $\mu = 1, \dots, k < n$. These constraints define a reduced configuration space $\mathcal{Q}_\mathcal{R}$ where we can define coordinates $Q = \{Q^a, a = 1, \dots, N := n - k\}$. The injective map $\mathcal{Q}_\mathcal{R} \rightarrow \mathcal{Q}$ is defined by some functions $q^i = q^i(Q^a)$ such that the rank of $|\partial q^i / \partial Q^a|$ is k (maximum rank), and the lifting of this map to the tangent structures allows to define the reduced

Lagrangian L_R as

$$L_R(Q, \dot{Q}) := L(q(Q), \frac{\partial q}{\partial Q} \dot{Q}).$$

Then, working in the second tangent bundles, it is easy to see that ³

$$[L_R]_a = [L]_i \frac{\partial q^i}{\partial Q^a},$$

where $[L]_i$ stand for the functional derivatives of L :

$$[L]_i := \frac{\partial L}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) =: \alpha_i - W_{ij} \ddot{q}^j.$$

Since $\partial f_\mu / \partial q^i$ form a basis for the independent null vectors of $\partial q^i / \partial Q^a$, i.e.,

$$\frac{\partial f_\mu}{\partial q^i} \frac{\partial q^i}{\partial Q^a} = 0 \tag{2.1}$$

identically, we can conclude that there exists some quantities λ^μ such that the equations $[L_R]_a = 0$ are equivalent to $[L]_i + \lambda^\mu \partial f_\mu / \partial q^i = 0$, $f_\nu(q) = 0$, which in turn are equivalent to the equations obtained from the variational principle for the extended lagrangian $L_E := L + \lambda^\mu f_\mu$, where λ^μ are taken as new dynamical variables.

So we have the well known result

$$[L_R] = 0 \iff [L_E] = 0,$$

which is independent on whether L is singular or not. Let us first consider the regular case. The evolution operator in $T\mathcal{Q}$ derived from L_E is

$$\mathbf{X} = \mathbf{X}_0 + \lambda^\mu \frac{\partial f_\mu}{\partial q^i} (W^{-1})^{ij} \frac{\partial}{\partial \dot{q}^j},$$

where

$$\mathbf{X}_0 = \dot{q}^i \frac{\partial}{\partial \dot{q}^i} + \alpha_i (W^{-1})^{ij} \frac{\partial}{\partial \dot{q}^j},$$

Stability of $\dot{f}_\nu := \mathbf{X} f_\nu = \mathbf{X}_0 f_\nu$ under \mathbf{X} determines the Lagrange multipliers as

$$\lambda^\nu = -\theta^{\mu\nu} \mathbf{X}_0 \dot{f}_\mu,$$

where $\theta^{\mu\nu}$ is the inverse of

³To alleviate the notation we sometimes do not write the pullbacks explicitly. Here for instance the pullback of the functions in $T^2\mathcal{Q}$ to functions in $T^2\mathcal{Q}_\mathcal{R}$ is understood.

$$\theta_{\mu\nu} = \frac{\partial f_\mu}{\partial q^i} (W^{-1})^{ij} \frac{\partial f_\nu}{\partial q^j}.$$

Observe that, except for the case when \dot{f}_μ is already stable under \mathbf{X}_0 , we end up with a dynamics which is different to the original one.

Now for the singular case. The equations of motion obtained from L are:

$$\alpha_i - W_{ij} \ddot{q}^j = 0.$$

Since W_{ij} is singular, it possesses r null vectors γ_ρ^i , giving up to r (independent or not) constraints

$$\alpha_i \gamma_\rho^i = 0.$$

It proves very convenient to use a basis for these null vectors which is provided from the knowledge of the r primary Hamiltonian constraints of the theory, ϕ_ρ^1 . Actually one can take [9]:

$$\gamma_\rho^i = \frac{\partial \phi_\rho^1}{\partial p_i}(q, \hat{p}), \quad (2.2)$$

where $\hat{p}_i(q, \dot{q}) = \partial L / \partial \dot{q}^i$. It is easily shown that there exists at least one M^{ij} and $\tilde{\gamma}_i^\rho$ such that

$$\delta_i^j = W_{is} M^{sj} + \tilde{\gamma}_i^\rho \gamma_\rho^j,$$

and therefore [10]

$$\ddot{q}^i = M^{is} \alpha_s + \tilde{\eta}^\rho \gamma_\rho^i,$$

where $\tilde{\eta}^\rho$ can be taken as arbitrary functions of t .

The stabilization algorithm starts by demanding that time evolution preserve the constraints $\alpha_i \gamma_\rho^i$. Sometimes new constraints are found; sometimes some of the $\tilde{\eta}^\rho$ are determined; eventually the dynamics is described by a vector field that exists on, and is tangent to, the constraint surface in velocity space:

$$\mathbf{X} := \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + a^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i} + \eta^\mu \Gamma_\mu =: \mathbf{X}_0 + \eta^\mu \Gamma_\mu;$$

the a^i are determined from the equations of motion and the stabilization algorithm; η^μ ($\mu = 1, \dots, p_1$) are arbitrary functions of time; and

$$\Gamma_\mu = \gamma_\mu^i \frac{\partial}{\partial \dot{q}^i},$$

where γ_μ^i are a subset of the null vectors of W_{ij} , corresponding to the first class primary constraints ϕ_μ ⁴ found in the Hamiltonian formalism [9] (which we take in number $r_1 \leq r$).

A holonomic GF for the dynamics will consist in the introduction of r_1 independent functions $f_\mu(q)$ in such a way that the arbitrary functions η^μ ($\mu = 1, \dots, r_1$) of the dynamics become determined by the requirement of stability of $\dot{f}_\nu := \mathbf{X}f_\nu = \mathbf{X}_0 f_\nu$ under the action of \mathbf{X} . To do so we need:

$$|\Gamma_\mu \dot{f}_\nu| = |\gamma_\mu^i \frac{\partial f_\nu}{\partial q^i}| = |\{\phi_\mu^1, f_\nu\}| =: |D_{\mu\nu}| \neq 0. \quad (2.3)$$

(observe that the relation $|\{f_\nu, \phi_\mu^1\}| \neq 0$ shows that the GF constraints $f_\mu = 0$ also fix the dynamics in canonical formalism, since ϕ_μ^1 are the first class primary constraints in the Hamiltonian formalism.).

With these GF constraints $f_\mu(q)$, the extended Lagrangian L_E gives the equations of motion:

$$[L]_i + \lambda^\mu \frac{\partial f_\mu}{\partial q^i} = 0, \quad f_\nu(q) = 0.$$

Now compute λ^μ . Contraction of γ_ν^i with the first set of these equations gives

$$\alpha_i \gamma_\nu^i + \lambda^\mu \gamma_\nu^i \frac{\partial f_\mu}{\partial q^i} = \alpha \gamma_\nu + \lambda^\mu D_{\nu\mu},$$

(we have suppressed some coordinate indices in the last expression) whereby we can get λ^μ as

$$\lambda^\mu = -(D^{-1})^{\mu\nu} \alpha \gamma_\nu. \quad (2.4)$$

The noticeable fact is that now the Legendre multipliers λ^μ are constraints. This is good news because it tells us that the introduction of the GF constraints has not modified the dynamics. Let us be more specific on this point.

There is a splitting of the set of hamiltonian primary constraints as first and second class (we will use indices μ for first class, and μ' for second class) constraints⁵. It will prove convenient later, since the first class constraints ϕ_μ^1 satisfy $|\{f_\nu, \phi_\mu^1\}| \neq 0$, to take a basis $\phi_{\mu'}^1$ for the second class constraints such that

$$\{\phi_{\mu'}^1, f_\nu\} = 0; \quad (2.5)$$

⁴Here we refer to *first class primary constraints* as the subset of primary constraints that still satisfy the first class condition *after* we have run the stabilization algorithm to get all the constraints of the theory: primary, secondary, etc. If we only look at the primary level, the number of first class primary constraints can be greater.

⁵The previous footnote applies here.

then

$$\gamma_{\mu'}^i \frac{\partial f_\nu}{\partial q^i} = 0 . \quad (2.6)$$

This splitting (μ, μ') is translated to the Lagrangian formalism through (2.2), so we end up with primary Lagrangian constraints $\alpha\gamma_\mu \simeq 0$ and $\alpha\gamma_{\mu'} \simeq 0$. It is worth noticing that only the first set is involved in the determination of λ^μ .

Now observe that, due to (2.4):

$$[L] + \lambda^\mu \frac{\partial f_\mu}{\partial q} = 0, \alpha\gamma_\mu = 0 \iff [L] = 0, \alpha\gamma_\mu = 0, \iff [L] = 0$$

the last equality holding because the constraints $\alpha\gamma_\mu = 0$ are consequence of $[L] = 0$. Therefore

$$\begin{aligned} [L] = 0, f_\mu(q) = 0 &\iff \\ [L] + \lambda^\mu \frac{\partial f_\mu}{\partial q} = 0, \alpha\gamma_\mu = 0, f_\mu(q) = 0 & \\ \iff [L_E] = 0, \alpha\gamma_\mu = 0 &\iff [L_R] = 0, \alpha\gamma_\mu = 0, \end{aligned} \quad (2.7)$$

where $\alpha\gamma_\mu$ is obviously understood with the pullback to $T\mathcal{Q}_R$, $i'^*(\alpha\gamma_\mu)$.

So we see that $[L] = 0, f_\mu(q) = 0 \implies [L_R] = 0$ but the converse is not true. In the next section we will get a perfect understanding of this fact.

3 When is L_R regular?

With the same notation as in the previous section, with L being a singular Lagrangian and L_R the reduced Lagrangian after an holonomic gauge fixing, we are going to prove the following

Theorem 1. *L_R is regular if and only if L has only first class (Hamiltonian) constraints.*

3.1 Proof of theorem 1

First observe that we can write

$$\frac{\partial^2 L_R}{\partial \dot{Q}^a \partial \dot{Q}^b} = W_{ij} \frac{\partial q^i}{\partial Q^a} \frac{\partial q^j}{\partial Q^b}, \quad (3.1)$$

where W_{ij} stands, as before, for $\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j$. We must check whether the Hessian matrix for L_R is regular or not. That is, look for the existence of solutions V^a of $(\partial^2 L_R / \partial \dot{Q}^a \partial \dot{Q}^b) V^b = 0$. Then, using (3.1):

$$0 = V^a \frac{\partial q^i}{\partial Q^a} \frac{\partial q^j}{\partial Q^b} W_{ij} = \frac{\partial q^j}{\partial Q^b} (W_{ij} \frac{\partial q^i}{\partial Q^a} V^a),$$

and since $\partial f_\mu / \partial q^j$ form a basis for the null vectors of $\partial q^j / \partial Q^b$, there must exists η^μ such that

$$W_{ij} \frac{\partial q^i}{\partial Q^a} V^a = \eta^\mu \frac{\partial f_\mu}{\partial q^j} ;$$

contraction with γ_ν^j , which are null vectors for W_{ij} , gives:

$$0 = \eta^\mu \gamma_\nu^j \frac{\partial f_\mu}{\partial q^j} = \eta^\mu D_{\nu\mu} ,$$

but since $|D_{\nu\mu}| \neq 0$ we conclude that $\eta^\mu = 0$, which means

$$W_{ij} \frac{\partial q^i}{\partial Q^a} V^a = 0 . \quad (3.2)$$

Now, since the set of null vectors of W_{ij} is γ_ν^j , $\gamma_{\nu'}^j$, from (3.2):

$$\frac{\partial q^i}{\partial Q^a} V^a = \delta^\nu \gamma_\nu^i + \delta^{\nu'} \gamma_{\nu'}^i ,$$

for some δ^ν , $\delta^{\nu'}$. Contraction of this last expression with $\partial f_\mu / \partial q^i$, and use of (2.1) and (2.6), gives:

$$\delta^\nu \gamma_\nu^i \frac{\partial f_\mu}{\partial q^i} = \delta^\nu D_{\nu\mu} = 0 \implies \delta^\nu = 0 ,$$

where (2.3) has also been used. Therefore

$$\frac{\partial q^i}{\partial Q^a} V^a = \delta^{\nu'} \gamma_{\nu'}^i ,$$

for $\delta^{\nu'}$ arbitrary. This means that we will get as many independent null vectors V for $\partial^2 L_R / \partial \dot{Q}^a \partial \dot{Q}^b$ as indices run for ν' , which is the number of second class primary Hamiltonian constraints. This proves the theorem, for if there are no second class primary constraints, there are no second class constraints at all. In such a case, there are no null vectors for $\partial^2 L_R / \partial \dot{Q}^a \partial \dot{Q}^b$ and L_R is regular.

To obtain a basis for the null vectors V^a is is convenient to have a deeper look at the canonical formalism.

3.2 canonical formalism

Here we will only consider some results that prove interesting to us. There is a natural map

$$\tilde{i} : T_{q(Q)}^* \mathcal{Q} \longrightarrow T_{\mathcal{Q}}^* \mathcal{Q}_{\mathcal{R}} .$$

Momenta p_i are mapped to momenta P_a of the reduced formalism according to $P_a = p_i \partial q^i / \partial Q^a$. Functions $\psi(q, p)$ in $T^* \mathcal{Q}$ are projectable to functions $\tilde{\psi}(Q, P)$ in $T^* \mathcal{Q}_{\mathcal{R}}$ if and only if

$$\psi(q(Q), p) = \tilde{\psi}(Q, p \frac{\partial q}{\partial Q}) \quad (3.3)$$

This projectability condition can be written in a more familiar way. From (3.3):

$$\frac{\partial \psi}{\partial p_i} = \frac{\partial \tilde{\psi}}{\partial P_a} \frac{\partial q^i}{\partial Q^a}.$$

Contraction with $\partial f_\mu / \partial q^i$, and use of (2.1) gives

$$\frac{\partial \psi}{\partial p_i} \frac{\partial f_\mu}{\partial q^i} = \{f_\mu, \psi\}|_{(f_\mu=0)} = 0.$$

This is the version of the projectability condition for ψ we were looking for:

$$\exists \tilde{\psi}, \tilde{i}^* : \psi \longrightarrow \tilde{\psi} \iff \{f_\mu, \psi\}|_{(f_\mu=0)} = 0 \quad (3.4)$$

So, according to (2.5), the second class primary constraints $\phi_{\mu'}^1$ are projectable to some functions $\tilde{\phi}_{\mu'}^1$. We will prove that $\tilde{\phi}_{\mu'}^1$ are the primary Hamiltonian constraints coming from the reduced Lagrangian L_R . First we can see that they are in the right number because it coincides with the number of null vectors V we have found for the Hessian matrix of L_R . Therefore we only have to check that the pullback of $\tilde{\phi}_{\mu'}^1$ to the tangent space $T\mathcal{Q}_{\mathcal{R}}$ is identically zero (this is the definition of primary constraints). In fact:

$$\tilde{\phi}_{\mu'}^1(Q, \frac{\partial L_R}{\partial \dot{Q}}) = \tilde{\phi}_{\mu'}^1(Q, \frac{\partial L}{\partial \dot{q}} \frac{\partial q}{\partial Q}) = \phi_{\mu'}^1(q(Q), \frac{\partial L}{\partial \dot{q}}) = 0$$

identically. We have thus got a basis for the null vectors V as $V_{\mu'}^a = \partial \tilde{\phi}_{\mu'}^1 / \partial P_a := \tilde{\gamma}_{\mu'}^a$.

Now we can get the primary Lagrangian constraints for L_R . It is easy to see that

$$\alpha_{Ra} := \frac{\partial L_R}{\partial Q^a} - \dot{Q}^b \frac{\partial^2 L_R}{\partial Q^b \partial \dot{Q}^a} = (\alpha_i + W_{ij} \dot{Q}^c \frac{\partial^2 q^j}{\partial Q^b \partial Q^c}) \frac{\partial q^i}{\partial Q^a} ;$$

then the primary Lagrangian constraints $\alpha_R \tilde{\gamma}_{\mu'}$ for L_R are

$$\begin{aligned} \alpha_{Ra} \tilde{\gamma}_{\mu'}^a &= (\alpha_i + W_{ij} \dot{Q}^c \frac{\partial^2 q^j}{\partial Q^b \partial Q^c}) \frac{\partial q^i}{\partial Q^a} \frac{\partial \tilde{\phi}_{\mu'}^1}{\partial P_a} = \\ &= (\alpha_i + W_{ij} \dot{Q}^c \frac{\partial^2 q^j}{\partial Q^b \partial Q^c}) \frac{\partial \phi_{\mu'}^1}{\partial p_i} = \alpha_i \frac{\partial \phi_{\mu'}^1}{\partial p_i} = \alpha \gamma_{\mu'}^i , \end{aligned}$$

where obvious pullbacks to $T\mathcal{Q}_{\mathcal{R}}$ are understood.

We now have a complete understanding of (2.7):

$$[L] = 0, f_\mu(q) = 0 \iff [L_R] = 0, \alpha\gamma_\mu = 0, \quad (3.5)$$

for the dynamics for L_R only provides with the primary constraints $\alpha_R\tilde{\gamma}_{\mu'}$, whereas the rest of primary constraints for L , $\alpha\gamma_\mu$, are absent in the reduced formulation, and must be separately introduced in order to maintain equivalence with $[L] = 0, f_\mu(q) = 0$.

Now we are ready to prove:

Theorem 2. *The dynamics derived from L_R has no gauge freedom.*

This is equivalent to say that all Hamiltonian constraints for L_R are second class.

3.3 Proof of theorem 2

First we will get the canonical Hamiltonian for L_R . The dynamics determined by the GF $f_\mu = 0$ is described in the canonical formalism by a first class Hamiltonian H_{FC} which can be taken to satisfy $\{f_\mu, H_{FC}\} = 0$ (Since the stabilization of $f_\mu = 0$ determines H_{FC}). According to (3.4), H_{FC} is projectable to a function $H_R := \tilde{H}_{FC}$ in $T^*\mathcal{Q}_R$

We can prove that H_R is a canonical Hamiltonian corresponding to L_R . First consider the fact, very easy to verify, that the Lagrangian energy $E_R := \dot{Q}\partial L_R/\partial\dot{Q} - L_R$ satisfies $E_R := i'^*(E)$, where E is the Lagrangian energy for L . Now, defining $FL : T\mathcal{Q} \longrightarrow T^*\mathcal{Q}$ as the Legendre map derived from L , and $FL_R : T\mathcal{Q}_R \longrightarrow T^*\mathcal{Q}_R$ as the Legendre map from L_R , then the following property holds: for any function ψ in $T^*\mathcal{Q}$ projectable to $\tilde{\psi}$ in $T^*\mathcal{Q}_R$, $FL_R^*\tilde{\psi} = i'^*FL^*\psi$. Then $E_R = i'^*(E) = i'^*FL^*H_{FC} = FL_R^*\tilde{H}_{FC}$, which proves that $H_R := \tilde{H}_{FC}$ is a good Hamiltonian for L_R .

Next, it is easy to see that for two projectable functions, ψ, ξ , (see (3.4)), its Poisson Bracket (PB) is projectable and satisfies:

$$\{\widetilde{\psi}, \widetilde{\xi}\} = \{\tilde{\psi}, \tilde{\xi}\}_R, \quad (3.6)$$

where $\{\cdot, \cdot\}_R$ stands for the PB in $T^*\mathcal{Q}_R$ (The proof of (3.6) is immediate).

Now we can realize that the stabilization algorithm for the second class primary constraints $\phi_{\mu'}$ in $T^*\mathcal{Q}$ is exactly the same as for the primary constraints $\tilde{\phi}_{\mu'}$ in $T^*\mathcal{Q}_R$. Indeed, the time derivative of $\tilde{\phi}_{\mu'}^1$ is:

$$\dot{\tilde{\phi}}_{\mu'}^1 = \{\tilde{\phi}_{\mu'}^1, H_R\}_R = \{\phi_{\mu'}^1, \widetilde{H_{FC}}\} =: \tilde{\phi}_{\mu'}^2,$$

and so on. But since the set of constraints $\phi_{\mu'}^1, \phi_{\mu'}^2, \dots$ (until the stabilization algorithm eventually ends when no new constraints appear) is second class, we conclude, using

(3.6) that so it must be for $\tilde{\phi}_{\mu'}^1, \tilde{\phi}_{\mu'}^2, \dots$. Hence Theorem 2 has been proved. Notice that H_R actually is a first class Hamiltonian.

The final picture for the reduction $L \rightarrow L_R$ through the GF constraints $f_\mu = 0$ has been established: In the reduced theory there is only place for the second class Hamiltonian constraints of the original theory. First class constraints have simply disappeared. Not only the primary ones ψ_μ^1 but also the secondary ψ_μ^2 , etc. that arise through the application of the stabilization algorithm to ψ_μ^1 . Consequently, the pullbacks of these first class constraints to velocity space, also disappear from the reduced velocity space. All the information carried by the first class structure has been erased. This applies in particular to the gauge generators, which are special combinations –with arbitrary functions and its time derivatives as coefficients– of the first class constraints.

4 Examples

4.1 Axial gauges in electromagnetism

In the introduction we have mentioned the example of electromagnetism in $n \cdot A = 0$ gauges. $n^\mu = (n^0, \mathbf{n})$, $A^\mu = (A^0, \mathbf{A})$. After elimination of A^0 the Lagrangian becomes regular. Gauss law is missing. But there is a compatibility between this missing constraint and the reduced dynamics. Now the time evolution vector field is tangent to the Gauss law, and if our initial (for, say, time $t=0$) configuration of the field \mathbf{A} satisfies $\nabla \mathbf{E} = 0$, then the constraint will be satisfied for all times. In QED we can use the reduced Lagrangian to write down the propagator for the photon in the gauge $n \cdot A = 0$, but to settle the asymptotic initial and final states, we must require the fulfillment of Gauss law [11].

Let $L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ be our starting point (Our metric is $(+, -, -, -)$). The only Lagrangian constraint is $\alpha\gamma = \partial_0(\partial_i A^i) + \Delta A_0 = 0$, the Gauss law. The time evolution operator, which only exists on, and is tangent to, the surface defined by the Gauss law, is:

$$\mathbf{X} = \int d^3y \dot{A}^\mu \frac{\delta}{\delta A^\mu(\mathbf{y})} - \int d^3y (\partial_j(\partial_\mu A^\mu) + \Delta A_j) \frac{\delta}{\delta \dot{A}^j(\mathbf{y})} + \int d^3y \lambda(\mathbf{y}, t) \frac{\delta}{\delta A^0(\mathbf{y})},$$

where λ is the arbitrary function of the dynamics reflecting the existence of gauge freedom. The GF $n \cdot A = 0$ will fix λ as

$$\lambda = -\frac{1}{n^0}(\mathbf{n} \cdot \nabla(\partial_\mu A^\mu) + \Delta(\mathbf{n} \cdot \mathbf{A})).$$

If the variable A_0 is eliminated by using the GF constraint, we end up with the reduced evolution operator –always tangent to the Gauss law constraint–

$$\hat{\mathbf{X}} = \int d^3y \dot{A}^j \frac{\delta}{\delta A^j(\mathbf{y})} - \int d^3y (\partial_j (\partial_i A^i + \frac{1}{n^0} \partial_0 (\mathbf{n} \cdot \mathbf{A})) + \Delta A_j) \frac{\delta}{\delta \dot{A}^j(\mathbf{y})}. \quad (4.1)$$

Now let us proceed the other way around, that is, instead of substituting the GF constraint $n \cdot A = 0$ into the equations on motion, let us plug $n \cdot A = 0$ into the Lagrangian L to get the reduced Lagrangian $L_R[\mathbf{A}, \dot{\mathbf{A}}]$:

$$L_R = -\frac{1}{4} F_{ij} F^{ij} - \frac{1}{2} F_{0i} F^{0i},$$

with $F_{0i} = \dot{A}_i - \frac{1}{n^0} \partial_i (\mathbf{n} \cdot \mathbf{A})$. This Lagrangian is regular and its time evolution operator is:

$$\begin{aligned} \mathbf{X}_R = & \int d^3y \dot{\mathbf{A}}^j \frac{\delta}{\delta \dot{\mathbf{A}}^j(\mathbf{y})} \\ & - \int d^3y \left(\Delta A_j + \frac{1}{n^0} (\partial_0 \partial_j + \frac{n^j}{n^0} \Delta) (\mathbf{n} \cdot \mathbf{A}) + (\partial_j + \frac{n^j}{n^0} \partial_0) (\partial_i A^i) \right) \frac{\delta}{\delta \dot{\mathbf{A}}^j(\mathbf{y})} \end{aligned} \quad (4.2)$$

As it was expected, there is no trace of the Gauss law. We can check, though, that the evolutionary vector field \mathbf{X}_R in (4.2) differs from the evolutionary vector field $\hat{\mathbf{X}}$ in (4.1) by a term which is proportional to the Gauss law. Indeed:

$$\hat{\mathbf{X}} - \mathbf{X}_R = \int d^3y \frac{n^j}{n^0} (\partial_0 (\partial_i A^i) + \frac{1}{n^0} \Delta (\mathbf{n} \cdot \mathbf{A})) \frac{\delta}{\delta \dot{\mathbf{A}}^j(\mathbf{y})}$$

This result explicitly exemplifies the relation (3.5).

4.2 Pure Abelian Chern-Simons

Consider the Abelian Chern-Simons 2+1 Lagrangian $L = \frac{1}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho$ (no metric involved). The primary hamiltonian constraints are

$$\phi^\sigma := \pi^\sigma - \epsilon^{0\sigma\rho} A_\rho = 0.$$

ϕ^0 is a first class constraint, whereas ϕ^1, ϕ^2 are second class. There is a secondary first class constraint $\psi := \partial_i (\pi^i + \epsilon^{0ij} A_j) = 0$ which plays the role of the Gauss law in this case.

Use of the GF constraint $A^0 = 0$ to get the reduced Lagrangian L_R gives:

$$L_R = \epsilon^{0ij} \dot{A}_i A_j,$$

which is still singular. Its only primary constraints are second class: $\tilde{\phi}^1 := \pi^1 - A_2 = 0$, $\tilde{\phi}^2 := \pi^2 + A_1 = 0$. No secondary constraints arise and Gauss law has disappeared. Observe that we are verifying our **theorem 2**: the reduced Lagrangian L_R is still singular (because L has second class constraints) but has no room for gauge freedom.

5 conclusions

In this paper we have proved some results concerning the correctness of plugging a set of holonomic GF constraints into a singular Lagrangian L . Our result is: if the GF constraints are taken in such a way that properly fix the dynamics of the singular theory defined by L , then the reduced Lagrangian L_R only keeps the information of the second class Hamiltonian constraints of the original theory. All first class constraints have disappeared from the reduced formalism. Therefore, to maintain equivalence with

the dynamics defined by L we must add the primary first class constraints to the dynamics defined by L_R . It is remarkable that all the information carried by the first class structure has been erased. Not only at the primary level, but at any level of the stabilization algorithm. This applies in particular to the gauge generators, which are made up of the first class constraints.

This is the picture in Hamiltonian formalism. In Lagrangian formalism the constraints erased through this procedure of plugging the GF constraints into the Lagrangian are just the pullbacks to velocity space of the first class Hamiltonian constraints of the original theory.

Lorentz covariance of Yang-Mills type theories requires the existence of secondary Hamiltonian constraints (whose pullbacks to velocity space will be part of the primary Lagrangian constraints). Also general covariance for theories containing more than scalar fields (like General Relativity) require these secondary constraints also. Therefore, in all these cases, the price for plugging the holonomic GF constraints into the Lagrangian will be the loss of the secondary first class constraints. The examples provided in the previous section not only show how this happens but also show the consistency of the reduced theory with the missing constraints.

In our paper we have dealt with holonomic GF constraints only. The advantage of considering this case is that we have to our avail a reduced configuration space out of which both the reduced velocity and phase space are built.

Let us finish with two more comments. First, the theory we have developed can be easily extended to cases where only a partial gauge fixing of the dynamics is performed. In such cases, there is still some gauge freedom left for L_R and only a part of the original primary first class constraints –and its descendants through the stabilization algorithm– disappear from the reduced formalism. Second, In the usual case (Yang-Mills, Einstein-Hilbert gravity, etc.) where the stabilization algorithm in velocity space has only one step, the surface defined by $\alpha\gamma_\mu = 0$ (see Eq. (3.5)) is a constant of motion for L_R , but not a standard one, since $\alpha\gamma_\mu = c \neq 0$ will not be, in general, a constant of motion. Our first example can be used to illustrate this aspect.

6 Acknowledgements

This work has been partially supported by the CICIT (project number AEN-0695) and by a Human Capital and Mobility Grant (ERB4050PL930544). The author thanks the Center for Relativity at The University of Texas at Austin for its hospitality.

References

- [1] H. Rund, “The Hamilton-Jacobi Theory in the Calculus of Variations”. D Van Nostrand Publ. Co., London, 1966.
- [2] E.J. Saletan and A.H. Cromer, *Am. J. Phys.* **38** (1970), 892.
- [3] E.J. Saletan and A.H. Cromer, “Theoretical Mechanics”. Wiley, New York, 1971.
- [4] J.F. Cariñena and M.F. Rañada, *J. Phys.A: Math. Gen.* **26** (1993), 1335.
- [5] P.A.M. Dirac, *Can. J. Math.* **2** (1950), 129.
- [6] P.A.M. Dirac, “Lectures on Quantum Mecanics”, New York: Yeshiva Univ. Press, 1964.
- [7] X. Gracia and J.M. Pons, *Annals of Physics (N.Y.)* **187** (1988), 2705.
- [8] J.M. Pons and L.C. Shepley, “Evolutionary Laws, Initial Conditions, and Gauge Fixing in Constrained Systems”. Preprint UB-ECM-PF 94/35. 1994.
- [9] C. Batlle, J. Gomis, J.M. Pons, and N. Roman, *J. Math. Phys.* **27** (1986), 2953.
- [10] K. Kamimura, *Nuovo Cimento* **B69** (1982), 33.
- [11] P.V. Landshoff and P. van Nieuwenhuizen, preprint DAMTP 93-33 (hep-th 9307117) (1993).